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Electromagnetic fields in coupled cavities.

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ELECTROMAGNETIC FIELDS IN COUPLED CAVITIES

BY

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ABSTRACT

Two cavities resonating at about the same frequency are coupled with each other through a small iris closed by a thin partition. The radius of this iris is assumed to be small compared with the wave-length. If one cavity is excited, fictitious surface magnetic charge and a fictitious surface magnetic current density are formed in the iris. The distribution of these magnetic charge and current densities constitutes a boundary condition for the calculation of the electromagnetic fields in the cavities. This solution is in terms of certain orthogonal functions of the separate cavities.

ACKNOWLEDGEMENTS

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1. INTRODUCTION

An electromagnetic field vector function, as defined by Helmhotz, can be represented by the sum of an irrotational function and a solenoidal function. This concept was first applied to the theory of cavity resonators by Slater [1]. However, he considered that only the electric field in the cavity consists of both irrotational as well as solenoidal fields; that the magnetic field, on the other hand, has no irrotational part. Later, Teichmann and Wigner [2] pointed out that a complete expansion of electromagnetic field should include the irrotational magnetic field which was found to be inversely proportional to its frequency Basing on this suggestion, Kurokawa [3] gave a proof that for a complete expansion of electromagnetic field it is necessary to add the irrotational component in the magnetic field. He asserted that if the irrotational component was neglected, there would be no magnetic field through the opening of the cavity at all when the cavity is coupled to a waveguide through an iris.

Cavity coupling systems have long been used and studied by physicists and engineers. Either in experimental or in theoretical analysis, they are generally considered as equivalent admittance or impedance circuit representation in form of parallel or series resonant circuits or both [4]. The coupling device is represented by an equivalent capacitive reactance if electrically coupled; by an equivalent inductive reactance if magnetically coupled [5]. This approach has little or no contribution to the analysis of the fields in the coupling systems. Up to present time, not much effort has been made on the field analysis as yet.

For most common practice, the cavities are coupled through apertures. The degree of coupling is depending on the size and shape of the aperture. If the aperture is large, there are many difficulties in dealing with the field analysis of cavity-coupling problem: large disturbance due to the presence of the aperture; strong interaction due to the infinite number of reflections of fields between cavities; and the unknown field distributions in the region of the aperture. On the other hand, if the aperture is small, all these field perturbations may be negligible and an approximation method may be applied in solving this problem. In this thesis, the author makes an attempt to work out the fields solution for two coupled cavities by combining the theory of cavity resonators and the Bethe's method of the field diffraction by small holes [6]. An example is also given to illustrate the application of the solution.

2. THE COMPLETE EXPANSION OF ELECTROMAGNETIC FIELDS IN CAVITY RESONATORS

Resonance phenomena may occur in a number of types of structures which are used in microwave system. Any volume completely enclosed by a conductor is usually called a cavity resonator. In distinction with a lumped constant resonant circuit, a cavity resonator can resonate at an infinite number of discrete frequencies. The electromagnetic fields in a cavity resonator can be represented by the sum of modes of certain frequencies of oscillation which are the characteristics of the eigenvalues of the wave equation with suitable boundary conditions. Both for electric field and magnetic field at an interior point in the cavity resonator, we have irrotational as well as solenoidal components.[1]

In the expansion for the electromagnetic fields in the cavity resonator, we can express the electric field $\vec{E}$ as a series of $\vec{E}_\alpha$ s and $\vec{F}_\alpha$ s, and the magnetic field $\vec{H}$ as a series of $\vec{H}_\alpha$ s and $\vec{G}_\beta$ s. That is,

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1 See Appendix A.

$$\vec{E} = \sum_a r_a \vec{E}_a + \sum_\alpha p_\alpha \vec{F}_\alpha \qquad \text{.........(2.1)}$$

$$\vec{H} = \sum_a s_a \vec{H}_a + \sum_\beta q_\beta \vec{G}_\beta \qquad \text{.........(2.2)}$$

where r_a, s_a, p_α and q_β are the time dependent expansion coefficients; $\vec{E}_a$, $\vec{H}_a$, $\vec{F}_\alpha$ and $\vec{G}_\beta$ are the functions of space. The orthogonal functions $\vec{E}_a$ s and $\vec{H}_a$ s are the solutions of wave equations with the corresponding boundary conditions.

$$\nabla^2 \vec{E}_a - k_a^2 \vec{E}_a = 0 \qquad \text{(in V)} \qquad \text{.........(2.3a)}$$

$$\vec{n} \times \vec{E}_a = 0 \qquad \text{(on S)} \qquad \text{.........(2.3b)}$$

$$\nabla^2 \vec{H}_a - k_a^2 \vec{H}_a = 0 \qquad \text{(in V)} \qquad \text{.........(2.4a)}$$

$$\vec{n} \cdot \vec{H}_a = 0 \qquad \text{(on S)} \qquad \text{.........(2.4b)}$$

The orthogonal functions $\vec{E}_a$ and $\vec{H}_a$ satisfy the relations:

$$k_a \vec{E}_a = \nabla \times \vec{H}_a \qquad \text{.........(2.5a)}$$

$$k_a \vec{H}_a = \nabla \times E_a \qquad \text{.........(2.5b)}$$

For the orthogonal functions $\vec{F}_\alpha$ and $\vec{G}_\beta$, we have

$$k_\alpha \vec{F}_\alpha = \nabla \psi_\alpha \qquad \text{.........(2.6a)}$$

$$\nabla^2 \psi_\alpha + k_\alpha^2 \psi_\alpha = 0 \qquad \text{(in V)} \quad \text{.........(2,6b)}$$

$$\psi_\alpha = 0 \qquad \text{(on S)} \quad \text{.........(2.6c)}$$

and

$$k_\beta \vec{G}_\beta = \nabla \phi_\beta \qquad \text{(in V)} \quad \text{.........(2.7a)}$$

$$\nabla^2 \phi_\beta + k_\beta^2 \phi_\beta = 0 \qquad \text{(in V)} \quad \text{.........(2.7b)}$$

$$\frac{\partial \phi_\beta}{\partial n} = 0 \qquad \text{(on S)} \quad \text{.........(2.7c)}$$

We assume that the medium in the volume of the cavity V enclosed by a perfectly conducting surface S is homogeneous and isotropic, that is, ϵ and μ are constant throughout the entire region. Thus Maxwell's equations are given in the form

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = -\vec{J}^* \qquad \text{.........(I)}$$

$$\nabla \times \vec{H} + \frac{\partial \vec{D}}{\partial t} = \vec{J} \qquad \text{.........(II)}$$

$$\nabla \cdot \vec{B} = \rho^* \qquad \text{.........(III)}$$

$$\nabla \cdot \vec{D} = \rho \qquad \text{.........(IV)}$$

The quantities $\vec{J}^*$ and ρ^* are fictitious volume densities of magnetic current and magnetic charge which have no physical existence. But they will contribute to the discontinuity in both $\vec{E}$ and $\vec{H}$. Currents and charges of both types satisfy the equations of continuity

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \qquad \text{.........(Va)}$$

$$\nabla \cdot \vec{J}^* + \frac{\partial \rho^*}{\partial t} = 0 \qquad \text{.........(Vb)}$$

By the use of the orthogonal property [2] of the functions of $\vec{E}_a$, $\vec{H}_a$, $\vec{F}_\alpha$, and $\vec{G}_\beta$, Eqs. (2.1) and (2.2) may be rewritten as:

$$\vec{E} = \sum_a \vec{E}_a \iiint_V \vec{E} \cdot \vec{E}_a \, dv + \sum_\alpha \vec{F}_\alpha \iiint_V \vec{E} \cdot \vec{F}_\alpha \, dv \qquad \text{.........(2.8)}$$

$$\vec{H} = \sum_a \vec{H}_a \iiint_V \vec{H} \cdot \vec{H}_a \, dv + \sum_\beta \vec{G}_\beta \iiint_V \vec{H} \cdot \vec{G}_\beta \, dv \qquad \text{.........(2.9)}$$

In addition to $\vec{E}$ and $\vec{H}$, we define

$$\vec{J}^* = \sum_a \vec{H}_a \iiint_V \vec{J}^* \cdot \vec{H}_a \, dv + \sum_\beta \vec{G}_\beta \iiint_V \vec{J}^* \cdot G_\beta \, dv \qquad \text{.........(2.10)}$$

$$\vec{J} = \sum_a \vec{E}_a \iiint_V \vec{J} \cdot \vec{E}_a \, dv + \sum_\alpha \vec{F}_\alpha \iiint_V \vec{J} \cdot \vec{F}_\alpha \, dv \qquad \text{.........(2.11)}$$

$$\rho^* = \sum_\beta \phi_\beta \iiint_V \rho^* \phi_\beta \, dv \qquad \text{.........(2.12)}$$

$$\rho = \sum_\alpha \psi_\alpha \iiint_V \rho \psi_\alpha \, dv \qquad \text{.........(2.13)}$$

The next step is to substitute these series expansions into the Maxwell's equations. From Eqs.(2.8) and (2.9),

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2 See Appendix B

we realize that the curl of $\vec{E}$ is equivalent to $\vec{H}$. that is,

$$\nabla \times \vec{E} = \sum_{\alpha} \vec{H}_\alpha \iiint_V \nabla \times \vec{E} \cdot \vec{H}_\alpha \, dv + \sum_{\beta} \vec{G}_\beta \iiint_V \nabla \times \vec{E} \cdot G_\beta \, dv \qquad \text{.........(2.14)}$$

Upon making use of the vector identities,

$$\begin{aligned} \nabla \times \vec{E} \cdot \vec{H}_\alpha &= \nabla \times \vec{H}_\alpha \cdot \vec{E} + \nabla \cdot (\vec{E} \times \vec{H}_\alpha) \\ &= k_\alpha \vec{E} \cdot \vec{E}_\alpha + \nabla \cdot (\vec{E} \times \vec{H}_\alpha) \end{aligned}$$

$$\begin{aligned} \nabla \times \vec{E} \cdot \vec{G}_\beta &= \nabla \times \vec{G}_\beta \cdot \vec{E} + \nabla \cdot (\vec{E} \times \vec{G}_\beta) \\ &= \nabla \cdot (\vec{E} \times \vec{G}_\beta) \end{aligned}$$

and the divergence theorem, Eq. (2.14) becomes

$$\begin{aligned} \nabla \times \vec{E} = &\sum_{\alpha} \vec{H}_\alpha \left(k_\alpha \iiint_V \vec{E} \cdot \vec{E}_\alpha \, dv + \iint_S \vec{n} \times \vec{E} \cdot \vec{H}_\alpha \, ds \right) \\ &+ \sum_{\beta} \vec{G}_\beta \iint_S \vec{n} \times \vec{E} \cdot \vec{G}_\beta \, ds \end{aligned} \qquad \text{.........(2.15)}$$

From the first Maxwell's equation, we have

$$\begin{aligned} &\sum_{\alpha} \vec{H}_\alpha \left(k_\alpha \iiint_V \vec{E} \cdot \vec{E}_\alpha \, dv + \iint_S \vec{n} \times \vec{E} \cdot \vec{H}_\alpha \, ds \right) + \sum_{\beta} \vec{G}_\beta \iint_S \vec{n} \times \vec{E} \cdot \vec{G}_\beta \, ds \\ &\quad + \mu \frac{d}{dt} \left\{ \sum_{\alpha} \vec{H}_\alpha \iiint_V \vec{H} \cdot \vec{H}_\alpha \, dv + \sum_{\beta} \vec{G}_\beta \iiint_V \vec{H} \cdot \vec{G}_\beta \, dv \right. \\ &= \sum_{\alpha} \vec{H}_\alpha \iiint_V \vec{J}^* \cdot \vec{H}_\alpha \, dv + \sum_{\beta} \vec{G}_\beta \iiint_V \vec{J}^* \cdot \vec{G}_\beta \, dv \end{aligned}$$

Multiplying both sides by $\vec{H}_\alpha$ and integrating over the entire region V, we get

$$k_a \iiint_V \vec{E} \cdot \vec{E}_a \, dv + \mu \frac{d}{dt} \iiint_V \vec{H} \cdot H_a \, dv$$

$$= \iiint_V \vec{J}^* \cdot \vec{H}_a \, dv - \iint_S \vec{n} \times \vec{E} \cdot \vec{H}_a \, ds \qquad \text{.........(2.16)}$$

Repeating that again with $\vec{G}_\beta$, we get

$$\mu \frac{d}{dt} \iiint_V \vec{H} \cdot \vec{G}_\beta \, dv = \iiint_V \vec{J}^* \cdot \vec{G}_\beta \, dv - \iint_S \vec{n} \times \vec{E} \cdot \vec{G}_\beta \, ds \qquad \text{.........(2.17)}$$

In the like manner in obtaining $\nabla \times \vec{E}$, $\nabla \times \vec{H}$ is

$$\nabla \times \cdot \vec{H} = \sum_a \vec{E}_a \iiint_V \nabla \times \vec{H} \cdot \vec{E}_a \, dv + \sum_\alpha \vec{F}_\alpha \iiint_V \nabla \times \vec{H} \cdot \vec{F}_\alpha \, dv$$

$$= \sum_a \vec{E}_a \left(k_a \iiint_V \vec{H} \cdot \vec{H}_a \, dv + \iint_S \vec{n} \times \vec{H} \cdot \vec{E}_a \, ds \right)$$

$$+ \sum_\alpha \vec{F}_\alpha \iint_S \vec{n} \times \vec{H} \cdot \vec{F}_\alpha \, ds \qquad \text{.........(2.18)}$$

Upon making use of the vector identities

$\vec{n} \times \vec{H} \cdot \vec{F}_\alpha = \vec{H} \cdot \vec{n} \times \vec{F}_\alpha$ and $\vec{n} \times H \cdot \vec{E}_a = \vec{H} \cdot \vec{n} \times \vec{E}_a$

and the boundary conditions $\vec{n} \times \vec{E}_a = 0$ and $\vec{n} \times \vec{F}_\alpha = 0$ on S, Eq. (2.18) reduces into

$$\nabla \times \vec{H} = \sum_a \vec{E}_a k_a \iiint_V \vec{H} \cdot \vec{H}_a \, dv \qquad \text{.........(2.19)}$$

We again substitute the known functions into the second Maxwell's equation and take advantage of their orthogonality as for the first Maxwell's equation. The final results are

$$k_a \iiint_V \vec{H} \cdot \vec{H}_a \, dv - \epsilon \frac{d}{dt} \iiint_V \vec{E} \cdot \vec{E}_a \, dv = \iiint_V \vec{J} \cdot \vec{E}_a \, dv \qquad \text{.........(2.20)}$$

$$-\epsilon \frac{d}{dt} \iiint_V \vec{E} \cdot \vec{F}_\alpha \, dv = \iiint_V \vec{J} \cdot \vec{F}_\alpha \, dv \qquad \text{.........(2.21)}$$

From the third Maxwell's equation, we obtain

$$\mu k_\beta \iiint_V \vec{H} \cdot \vec{G}_\beta \, dv = \iiint_V \rho^* \phi_\beta \, dv - \mu \iint_S \vec{H} \cdot \vec{n} \, \phi_\beta \, ds \qquad \text{.........(2.22)}$$

Finally, the forth Maxwell's equation gives

$$-\epsilon k_\alpha \iiint_V \vec{E} \cdot \vec{F}_\alpha \, dv = \iiint_V \rho \, \psi_\alpha \, dv \qquad \text{.........(2.23)}$$

Basing on that the electromagnetic fields must satisfy the Maxwell's equations, we have established Eqs. (2.16), (2.17), (2.20) , (2.21), (2.22), and (2.23) in solving the expansion coefficients r_a, s_a, p_α, and q_β as defined in Eqs. (2.1) and (2.2). As soon as sufficient conditions are given, these coefficients can be evaluated. And the desired electromagnetic fields in a cavity resonator will be obtained in series expansion.

3. ASSUMPTIONS AND BOUNDARY CONDITIONS

Generally, the cavities communicate with one another by means of apertures or irises in their common walls. The coupling between any one pair of cavities is dependent upon the size, the shape and the location of the iris. If the location is fixed, the coupling will be proportional to the size of the iris. Since the cavity resonator is a very sensitive device, a hole on the wall will shift the original resonant frequency and perturb the fields in amplitude, phase and orientation. The bigger the iris, the larger the change. Gould and Cunliffe [7] related the fields in two coupled cavities in such a way that the tangential component of electric field and magnetic field are continuous over the iris. They ignored the normal components of the electric and magnetic fields which also have the same contribution to the solution as the tangential components. The distribution of the fields over the region of the iris, however, must be given as a basic requirement to seek the solution for the field configurations in both cavities. Due to the fact that there is continuity of fields only in the iris and the field interaction between the two cavities occurs through the iris, the field distributions over the iris will be resultant of infinite number of incident and reflecting waves flowing via the iris. We need the boundary conditions ~~to solve~~ the coupling problem. In this case, however, we have to know the coupling before we can determine the boundary conditions. Under this condition, the solution seems hopeless.

Let us now imagine that the iris is so small that the interaction between these two coupled cavities can be made negligible. The fields in the iris, as pointed out by Bethe, are the incident fields from the main cavity which is originally excited. Since the energy is radiated via the iris, the iris will be considered as the source of excitation for the subsidiary cavity. The field configurations in the subsidiary cavity and the perturbed fields in the main cavity can be calculated in terms of fictitious magnetic charge and current density in the iris.

In solving the problem of two cavities coupled through a small iris, we make the following assumptions:

(1) The iris is circular and its radius is small compared with the wavelength.

(2) The thickness of the screen of the iris is

negligible.

(3) The screen of the iris is a plane surface.

(4) Both cavities contain the same medium which is assumed to be isotropic and homogeneous.

(5) No electric charge and current are present in the subsidiary cavity.

(6) The subsidiary cavity is originally unexcited.

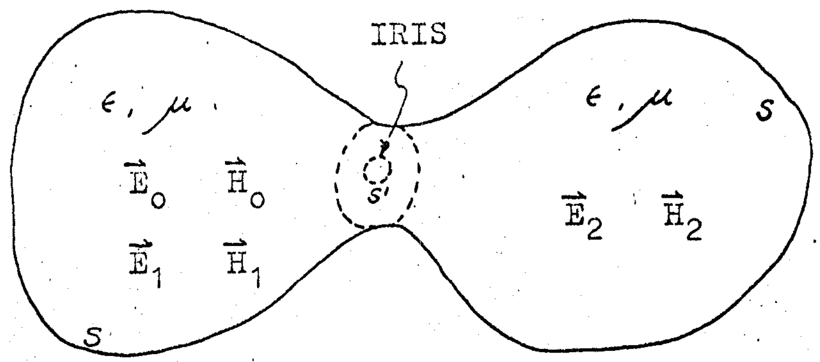

Fig. 1 Two cavities coupled by a circular iris

Let us consider two cavities of arbitrary shape coupled through a circular iris of radius a, as shown in Fig. 1. The main cavity is initially excited. Regardless of the source of excitation, the electromagnetic fields existing in this cavity may be expressed, in general, as:

$$\vec{E}_o = \sum_a r_{ao} \vec{E}_{ao} + \sum_\alpha p_{\alpha o} \vec{F}_{\alpha o} \qquad \text{.........(3.1)}$$

$$\vec{H}_o = \sum_a s_{ao} \vec{H}_{ao} + \sum_\beta g_{\beta o} \vec{G}_{\beta o} \qquad \text{.........(3.2)}$$

with the boundary conditions

$$\left.\begin{array}{r} \vec{n} \times \vec{E}_a = 0 \\ \vec{n} \cdot \vec{H}_a = 0 \end{array}\right\} \text{ on S}$$

$$\left.\begin{array}{r} \psi_\alpha = 0 \\ \partial\phi_\beta/\partial n = 0 \end{array}\right\} \text{ on S}$$

The fields incident upon the region of the iris when the latter is absent, as shown in Fig. 2a, are

$$E_{on} = \vec{n}_r \cdot \vec{E}_o = \sum_a r_{ao} \vec{n}_r \cdot \vec{E}_a + \sum_\alpha p_{\alpha o} \vec{n}_r \cdot \vec{F}_{\alpha o} \qquad \text{.........(3.3)}$$

$$\vec{H}_{ot} = \vec{n}_r \times \vec{H}_o = \sum_a s_{ao} \vec{n}_r \times \vec{H}_{ao} + \sum_\beta g_{\beta o} \vec{n}_r \times \vec{G}_\beta \qquad \text{.........(3.4)}$$

where $\vec{n}$ is the unit normal vector of the iris. In the presence of the iris, the boundary conditions at the iris as used by Bethe are as follows: the tangential component of magnetic field and the normal component of electric field are discontinuous. The iris is assumed to be small compared with the wavelength so that $\vec{n}_r \cdot \vec{E}_o$ and $\vec{n}_r \times \vec{H}_o$ may be considered constant over the iris and equal to their values at the center of the iris. We should note that this approximation would not be valid for a large iris. The total fields in the main cavity are $\vec{H}_o + \vec{H}_1$ and $\vec{E}_o + \vec{E}_1$. Let us denote the fields in the subsidary cavity are $\vec{E}_2$ and $\vec{H}_2$. $\vec{H}_o$ and $\vec{E}_o$ are the unperturbed fields and the $\vec{H}_1$ and $\vec{E}_1$ are the reflected fields in the main cavity due to the presence of the iris. $\vec{E}_2$ and $\vec{H}_2$ are the diffracted fields in the subsidary cavity. These fields satisfy the boundary conditions

$$\vec{H}_{2t} - \vec{H}_{1t} = \vec{H}_{ot} \qquad \text{.........(3.5)}$$

$$\vec{H}_{1t} + \vec{H}_{2t} = 0 \qquad \text{.........(3.6)}$$

$$\vec{H}_{1n} - \vec{H}_{2n} = 0 \qquad \text{.........(3.7)}$$

$$\vec{E}_{2n} - \vec{E}_{1n} = \vec{E}_{on} \qquad \text{.........(3.8)}$$

$$\vec{E}_{1n} + \vec{E}_{2n} = 0 \qquad \text{.........(3.9)}$$

$$\vec{E}_{1t} - \vec{E}_{2t} = 0 \qquad \text{.........(3.10)}$$

The above boundary conditions are clearly shown in Fig. 2b and Fig. 2c. From Eqs. (3.5) and (3.6), we find:

$$\vec{H}_{2t} = \tfrac{1}{2}\vec{H}_{ot} \text{ in the iris} \qquad \text{.........(3.11)}$$

From Eqs. (3.8) and (3.9), we have:

$$\vec{E}_{2n} = \tfrac{1}{2}\vec{E}_{on} \text{ in the iris} \qquad \text{.........(3.12)}$$

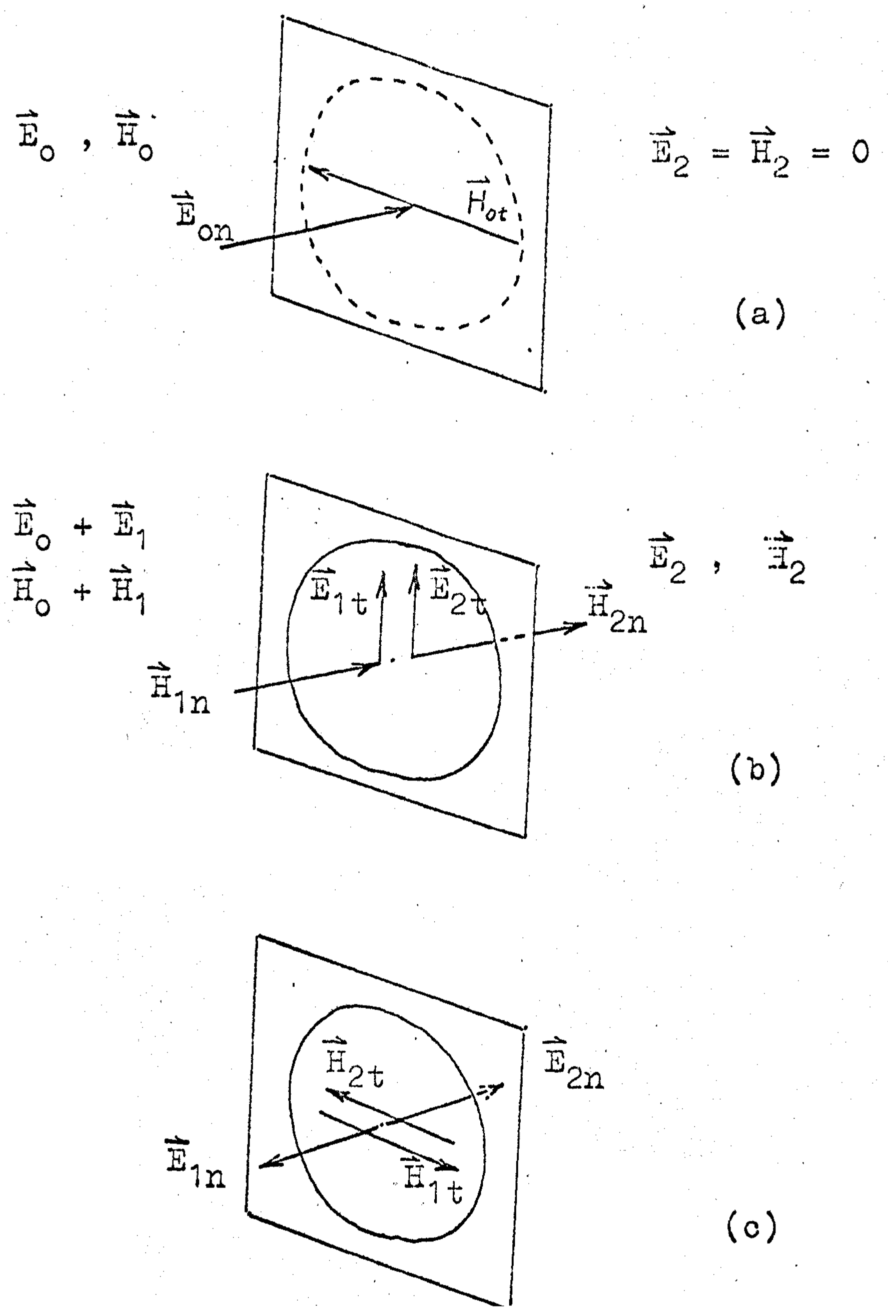

Fig. 2 Boundary conditions at the vicinity of the iris. (a) Before the iris is cut. (b) and (c) Reflected and diffracted fields after the iris is formed.

4. SURFACE MAGNETIC CHARGE & CURRENT DISTRIBUTIONS

In dealing with the discontinuities in electromagnetic fields which is physically impossible, it is usually explained by assuming the presence of surface distributions of magnetic charge and current whereby these discontinuities would be generated. In other words, the surface magnetic charge and current distributions may be induced by the discontinuities in both E and H. These quantities have no physical meaning but are mathematically important. For the same reason, the discontuities of the normal component of the electric field and tangential component of the magnetic field can be best analyzed by assuming the formation of fictitious surface magnetic charge and current densities in the coupling iris. Since the subsidiary cavity is initially unexcited, they are considered to be a complete set of the source of excitation from which the fields in that cavity originates. Maxwell's equations in three dimensions are given by:

$$\nabla \cdot \vec{H} = \rho^* \qquad \text{.........(4.1)}$$

$$\nabla \times \vec{E} + \mu \frac{\partial \vec{H}}{\partial t} = -\vec{J}^* \qquad \text{.........(4.2)}$$

and the continuity eqyation is given by

$$\nabla \cdot \vec{J}^* + \frac{\partial \rho^*}{\partial t} = 0 \qquad \text{.........(4.3)}$$

where $\vec{J}^*$ and ρ^* are the fictitious volume magnetic current density and charge density respectively. Let us now designate the fictitious surface charge density by η^* and the fictitious surface magnetic current
by η and t
By analogy to Eq. (4.3), $\vec{K}^*$ and η^* satisfy the relation

$$\nabla \cdot \vec{K}^* + \frac{\partial \eta^*}{\partial t} = 0 \qquad \text{.........(4.4)}$$

If either $\vec{K}^*$ or η^* is determined, the other one can be calculted by Eq. (4.4).

If there are no electric currents and charges present, the electromagnetic fields in the subsidiary cavity will satisfy Maxwell's equations

$$\nabla \times \vec{H}_2 - \epsilon \frac{\partial \vec{E}_2}{\partial t} = 0 \qquad \text{.........(4.5)}$$

$$\nabla \cdot \vec{E}_2 = 0 \qquad \text{.........(4.6)}$$

We assume now a scalar potential ϕ and a vector potential $\vec{A}$ so that

$$\vec{E}_2 = \nabla \times \vec{A} \qquad \text{.........(4.7)}$$

$$\vec{H}_2 = \frac{\partial A}{\partial t} - \nabla \phi \qquad \text{.........(4.8)}$$

as implied by Eqs. (4.5) and (4.6). Analogous to the electric case, the scalar and the vector potentials can be expressed by:

$$\phi(\vec{r}) = \frac{1}{4\pi} \iint_{S'} \eta^{*}(\vec{r}')\, g\,|\vec{r} - \vec{r}'|\, ds \qquad \text{.........(4.9)}$$

$$\vec{A}(\vec{r}) = \frac{1}{4\pi} \iint_{S'} \vec{K}^{*}(\vec{r}')\, g\,|\vec{r} - \vec{r}'|\, ds \qquad \text{.........(4.10)}$$

where $\vec{r}'$ is a source point on the iris; $\vec{r}$ is any vector in the cavity from the center of the iris; S' is the area of the iris; and $g\,|\vec{r} - \vec{r}'|$ is the Green's function

$$g\,|\vec{r} - \vec{r}'| = \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r} - \vec{r}'|} .$$

By substituting Eqs. (4.9) and (4.10) into Eqs. (4.7) and (4.8) and making use of the boundary conditions of Eqs. (3.11) and (3.12) as obtained in Section 3, we have

$$\vec{E}_{2n} = \frac{1}{2}\vec{E}_{on}$$

$$= \nabla_r \times \vec{A}(\vec{r}_h)$$

$$= \frac{1}{4\pi}\iint_{S'} K^*(\vec{r}') \times \nabla_r g|\vec{r}-\vec{r}'|\, ds \qquad \text{.........(4.11)}$$

$$\vec{H}_{2t} = \frac{1}{2}\vec{H}_{ot}$$

$$= \frac{\partial \vec{A}}{\partial t} - \nabla\phi$$

$$= \frac{1}{4\pi}\iint_{S'} \frac{\partial \vec{K}^*(\vec{r}')}{\partial t} g|\vec{r}-\vec{r}'|\, ds - \frac{1}{4\pi}\iint_{S'} \eta^*(\vec{r}')\nabla_r g|\vec{r}-\vec{r}'|\, ds \,. \qquad \text{.........(4.12)}$$

where $\vec{r}_h$ is the field point in the iris, and $\nabla_r = \vec{a}_r \dfrac{\partial}{\partial r}$.

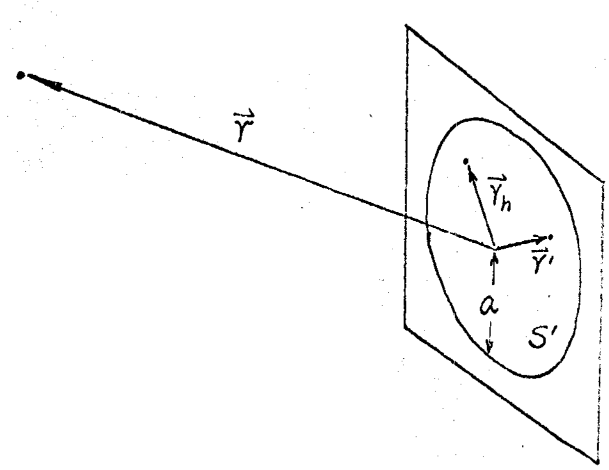

Fig. 3 Orientation of $\vec{r}$, $\vec{r}'$, and $\vec{r}_h$.

In Eq.(4.12), the integrand of the first term on the right hand side is of the order of a^2, while the second term is of the order of a^{-2}. Since the radius of the iris is small compared with the wavelength, we may neglect the first term of Eq. (4.12). Moreover, if we also neglect the retardation in $g\,|\vec{r} - \vec{r}'|$, Eq. (4.12) reduces to

$$\frac{1}{2}\vec{H}_{ot} = -\frac{1}{4\pi}\iint_{S'} \eta^*(\vec{r}')\,\nabla_r\, g'|\vec{r}_h - \vec{r}'|\,ds \quad \text{.........(4.13)}$$

where $g'\,|\vec{r}_h - \vec{r}'| = 1 / |\vec{r}_h - \vec{r}'|$

As an approximation, we may assume $\vec{H}_{2t}$ to be constant over the iris and equal to $\frac{1}{2}\vec{H}_{ot}$, Eq. (4.13) becomes

$$\frac{1}{2}\vec{H}_{ot}\cdot\vec{r}_h = -\frac{1}{4\pi}\iint_{S'} \eta^*(\vec{r}')\, g'|\vec{r}_h - \vec{r}'|\,ds \quad \text{.........(4.14)}$$

It is well known that a constant electric field may be produced by a uniform distribution of electric dipoles on an ellipsoid, the dipoles having the same direction as the field. The surface electric charge density will be proportional to the product of the field and the gradient of the surface electric dipole density. Let us apply the same idea to the magnetic field. The desired surface magnetic charge density in the iris is

$$\eta^* = -C\,\frac{\vec{H}_{ot}\cdot\vec{r}_h}{(a^2 - r_1^2)^{1/2}} \quad \text{.........(4.15)}$$

Upon substituting Eq. (4.15) into Eq. (4.14) and carrying out the integration, we can find that the proportionality constant C is

$$C = 1 / \pi^2$$

Consequently, the surface magnetic charge density is

$$\eta^* = - \frac{\vec{H}_{ot} \cdot \vec{r}_{h}}{\pi^2 (a^2 - r'^2)^{1/2}} \quad \text{.........(4.16)}$$

From the continuity equation of Eq. (4.4), the surface magnetic current is found to be

$$\vec{K}^* = - \frac{1}{\pi^2} (a^2 - r'^2)^{1/2} \frac{\partial \vec{H}_{ot}}{\partial t} \cdot \vec{r}' \quad \text{.........(4.17)}$$

Bethe chooses a magnetic charge density on the iris to create the proper $\vec{H}_{2t}$. The surface magnetic current density $\vec{K}^*$ creates a term in Eq. (4.11) of the order of a^2. Since this is presumably negligible as compared with the actual normal component of $\vec{E}_{2n} = \frac{1}{2} \vec{E}_{on}$, another magnetic current $\vec{K}'^*$ is needed to maintain the proper boundary condition for $\vec{E}_{2n}$ on the iris without disturbing the proper $\vec{H}_{2t}$ given by Eq. (4.12). This is accomplished by means of a circulating current with no divergence and hence no additional surface magnetic charge density. Using the same technique as in determining the magnetic charge density, this additional magnetic current density is obtained as

$$\vec{K}'^{*} = \frac{\vec{\gamma}' \times E_{on}}{2\pi^2 (a^2 - r'^2)^{1/2}} \,. \qquad \text{.........(4.18)}$$

In combining with Eq. (4.17), the total surface magnetic current density is

$$\vec{K}_T^{*} = \vec{K}^{*} + \vec{K}'^{*}$$

$$= -\frac{1}{\pi^2}\left[(a^2 - r'^2)^{1/2} \frac{\partial \vec{H}_{ot}}{\partial t} \cdot \vec{\gamma}' - \frac{\vec{\gamma}' \times \vec{E}_{on}}{2(a^2 - r'^2)^{1/2}} \right] \qquad \text{.........(4.19)}$$

Since the tangential component of the magnetic field and the normal component of the electric field over the iris are given as

$$\vec{H}_{ot} = \sum_a S_{ao}\, \vec{n}_r \times \vec{H}_{ao} + \sum_\beta g_\beta\, \vec{n}_r \times \vec{G}_\beta$$

$$\vec{E}_{on} = \sum_a \gamma_{ao}\, \vec{n}_r (\vec{n}_r \cdot \vec{E}_{ao}) + \sum_\alpha p_{\alpha o}\, \vec{n}_r (\vec{n}_r \cdot \vec{F}_\alpha)$$

Consequently, Eq. (4.19) and Eq. (4.16) can be written as

$$\vec{K}_T^{*} = -\frac{1}{\pi^2}\left[(a^2 - r'^2)^{1/2} \left\{ \sum_a \frac{\partial S_{ao}}{\partial t} \vec{n}_r \times \vec{H}_{ao} + \sum_\beta \frac{\partial g_{\beta o}}{\partial t} \vec{n}_r \times \vec{G}_\beta \right\} \right.$$

$$\left. - \frac{\vec{\gamma}' \times \vec{n}_r}{2(a^2 - r'^2)^{1/2}} \left\{ \sum_\alpha \gamma_{ao} (\vec{n}_r \cdot \vec{E}_{ao}) + \sum_\alpha p_{\alpha o}\, \vec{n}_r \cdot \vec{F}_{\alpha o} \right\} \right] \qquad \text{.........(4.20)}$$

$$\eta^{*} = -\frac{\vec{\gamma}'}{\pi^2 (a^2 - r'^2)^{1/2}} \cdot \left\{ \sum_a S_{ao}\, \vec{n}_r \times H_{ao} + \sum_\beta \frac{\partial g_{\beta o}}{\partial t} \vec{n}_r \times \vec{G}_{\beta o} \right\} \qquad \text{.........(4.21)}$$

5. GENERAL FIELD SOLUTION

The surface magnetic charge and current density distribution over the coupling iris was obtained in previous section. The next step is to determine the electromagnetic fields in the cavity coupling system. As stated before, the surface magnetic charge and current densities may be considered as the only sources of the excitation for the electromagnetic fields in the subsidiary cavity. The fields in the main cavity would not be the original fields alone but are the resultant of the original wave and the reflected wave due to the opening of the iris on the wall. In Section 4, we defined a scalar and a vector potential

$$\phi(\vec{r}, t) = \frac{1}{4\pi} \iint_{S'} \eta^*(\vec{r}') \, g|\vec{r} - \vec{r}'| \, ds \qquad \text{.........(5.1)}$$

$$\vec{A}(\vec{r}, t) = \frac{1}{4\pi} \iint_{S'} \vec{K}_T^* \, g|\vec{r} - \vec{r}'| \, ds \qquad \text{.........(5.2)}$$

Substituting the values of η^* and $\vec{K}_T^*$ (Eqs. (4.21) and (4.20)) into Eqs. (5.1) and (5.2) respectively, we have

$$\phi(\vec{r},t) = -\frac{1}{4\pi^3}\Big[\sum_a \frac{\partial S_{ao}}{\partial t}\iint_{S'} \frac{\vec{n}_r \times \vec{H}_{ao}\cdot\vec{r}'\, g\,|\vec{r}-\vec{r}'|}{(a^2 - r'^2)^{1/2}}\, ds$$

$$+ \sum_\beta \frac{\partial q_{\beta o}}{\partial t}\iint_{S'} \frac{\vec{n}_r \times \vec{G}_{\beta o}\cdot\vec{r}'\, g\,|\vec{r}-\vec{r}'|}{(a^2 - r'^2)^{1/2}}\, ds\Big]$$

$$\text{.........(5.3)}$$

$$A(\vec{r},t) = -\frac{1}{4\pi^3}\Big\{\Big[\sum_a \frac{\partial S_{ao}}{\partial t}\iint_{S'} \vec{n}_r \times \vec{H}_{ao}\,(a^2 - r'^2)^{1/2} g\,|\vec{r}-\vec{r}'|\, ds$$

$$+ \sum_\beta \frac{\partial q_{\beta o}}{\partial t}\iint_{S'} \vec{n}_r \times \vec{G}_{\beta o}\,(a^2 - r'^2)^{1/2} g\,|\vec{r}-\vec{r}'|\, ds\Big]$$

$$- \frac{1}{2}\Big[\sum_a r_{ao}\iint_{S'} \frac{\vec{r}'\times\vec{n}_r(\vec{n}_r\cdot\vec{E}_{ao})}{(a^2 - r'^2)^{1/2}}\, g\,|\vec{r}-\vec{r}'|\, ds$$

$$+ \sum_\alpha p_{\alpha o}\iint_{S'} \frac{\vec{r}'\times\vec{n}_r(\vec{n}_r\cdot\vec{F}_{\alpha o})}{(a^2 - r'^2)^{1/2}}\, g\,|\vec{r}-\vec{r}'|\, ds$$

$$\text{.........(5.4)}$$

Both Eq. (5.3) and Eq. (5.4) satisfy the Maxwell's equations

$$E_2(\vec{r},t) = \nabla \times \vec{A}(\vec{r},t) \qquad \text{.........(5.5)}$$

$$\vec{H}_2(\vec{r},t) = \frac{\partial \vec{A}(\vec{r},t)}{\partial t} - \nabla\phi(\vec{r},t) \qquad \text{.........(5.6)}$$

It is seen that Eq. (5.5) and Eq. (5.6) are similar in form to the general fields equations for a cavity resonator

$$\vec{E} = \sum_a r_a \vec{E}_a + \sum_\alpha p_\alpha \vec{F}_\alpha ,$$

$$\vec{H} = \sum_a s_a \vec{H}_a + \sum_\beta q_\beta \vec{G}_\beta .$$

We write the scalar potential as a product of a function of time and a function of space:

$$\phi(\vec{r},t) = \sum_\beta q_{\beta_2}(t)\, \phi_{\beta_2}(\vec{r}) . \qquad \text{.........(5.7)}$$

Similarly, the vector potential:

$$\vec{A}(\vec{r},t) = \sum_a r_{a_2}(t) \vec{A}_{a_2}(\vec{r}) \qquad \text{.........(5.8)}$$

Substituting Eqs. (5.7) and (5.8) into Eqs. (5.5) and (5.6) respectively, we have

$$\vec{E}_2(\vec{r},t) = \sum_a r_{a2} \nabla \times \vec{A}'_{a_2} \qquad \text{.........(5.9)}$$

$$\vec{H}_2(\vec{r},t) = \sum_a \frac{\partial r_{a_2}}{\partial t} \vec{A}'_{a_2} - \sum_\beta q_{\beta_2} \nabla \phi'_{\beta_2} \qquad \text{.........(5.10)}$$

where ϕ'_{β_2} and $\vec{A}'_{a_2}$ are the eigenfunctions satisfying the boundary conditions

$$\left.\begin{aligned} n \cdot \vec{A}_{a_2} &= 0 \\ \frac{\partial \phi_{\beta_2}}{\partial t} &= 0 \end{aligned}\right\} \text{ on S}$$

From Eq. (5.9), we see that the irrotational electric field does not appear in the expression. It is now necessary to give the proof that the exclusion of $\vec{F}_\alpha$ terms from the general field equations for the subsidiary cavity is justified. We recall that the surface magnetic charge and current densities over the iris are the only sources of excitation of the subsidiary cavity, and that there is no electric current or charge present in V. From the Maxwell's equation

$$\nabla \cdot \vec{D} = \rho = 0$$

the time function or the expansion coefficient of F_α is

$$\begin{aligned} p_\alpha &= \iiint_V \vec{E} \cdot \vec{F}_\alpha \, dv \\ &= -\frac{1}{k_\alpha \epsilon} \iiint_V \rho \, \psi_\alpha \, dv \\ &= 0 \end{aligned}$$

In Section 2, Eq. (2.22) gives

$$\begin{aligned} q_{\beta z} &= \iiint_V \vec{H} \cdot \vec{G}_{\beta z} \, dv \\ &= \frac{1}{\mu k_{\beta z}} \left[\iiint_V \rho^* \phi'_{\beta z} \, dv - \iiint_{S+S'} \vec{B} \cdot \vec{n} \, \phi'_{\beta z} \, ds \right] \end{aligned}$$

If $\rho^* = 0$ in region V,

$$q_{\beta z} = -\frac{1}{\mu k_{\beta z}} \iint_{S+S'} \vec{B} \cdot \vec{n} \, \phi'_{\beta z} \, ds .$$

Since $\vec{B}\cdot\vec{n} = 0$ on S and $\vec{B}\cdot\vec{n}_r = 0$ on S',

$$q_{\beta z} = -\frac{1}{\mu k_{\beta z}} \iint_{S'} \eta^*(\vec{r'})\, \phi'_{\beta z}(\vec{r}_h)\, ds \qquad \text{.........(5.11)}$$

By replacing η^* by its known value (Eq. (4.21)), Eq. (5.11) becomes

$$q_{\beta z} = \frac{1}{\pi^2 \mu k_{\beta z}} \Big[\sum_a S_{a0} \iint_{S'} \frac{\vec{r'}\cdot\vec{n}_r \times \vec{H}_{a0}\, \phi'_{\beta z}}{(a^2 - r'^2)^{1/2}}\, ds + \sum_\beta q_{\beta 0} \iint_{S'} \frac{\vec{r'}\cdot\vec{n}_r \times \vec{G}_{\beta 0}\, \phi'_{\beta z}}{(a^2 - r'^2)^{1/2}}\, ds \Big] \qquad \text{.........(5.12)}$$

Again from Eq. (2.16) in Section 2, we have

$$\mu\epsilon \frac{\partial^2 r_{az}}{\partial t^2} + k_{az}^2 r_{az} = \iiint_V \vec{J}^* \cdot \vec{H}_{az}\, dv - \iint_{S+S'} \vec{n} \times \vec{E} \cdot \vec{H}_{az}\, ds \qquad \text{..........(5.13)}$$

With the aid of the conditions:

$\vec{J}^* = 0$ in V

$\vec{n} \times \vec{E} = 0$ on S

$\vec{n}_r \times \vec{E} = \vec{K}_T^*$, on S',

Eq. (5.13) can be reduced to:

$$\mu\epsilon \frac{\partial^2 r_{az}}{\partial t^2} + k_{az}^2 r_{az} = -\iint_{S'} \vec{K}_T^* \cdot \vec{H}_{az}\, ds$$

For $\vec{H}_{a_z}(r) = \vec{A}'_{a_z}(r)$, we finally get

$$\mu\epsilon \frac{\partial^2 r_{a_z}}{\partial t^2} + k_{a_z}^2 r_{a_z} = -\iint_{S'} \vec{K}_T^* \cdot \vec{A}'_{a_z}(\vec{r}_h)\, ds \qquad \text{.........(5.14a)}$$

Since the value of $\vec{K}_T^*$ is given by Eq.(4.20) in Section 4, the right hand side of Eq. (5.14a) may be expanded into

$$- \iint_{S'} \vec{K}_T^* \cdot \vec{A}'_{a_z}(r_h)\, ds$$

$$= \frac{1}{\pi^2} \Big[\sum_a \frac{\partial r_{a_0}}{\partial t} \iint_{S'} \vec{n}_r \times \vec{H}_{a_0} \cdot \vec{A}'_{a_z} (a^2 - r'^2)^{1/2}\, ds$$

$$+ \sum_\beta \frac{\partial q_{\beta_0}}{\partial t} \iint_{S'} \vec{n}_r \times \vec{G}_{\beta_0} \cdot \vec{A}'_{a_z} (a^2 - r'^2)^{1/2}\, ds$$

$$- \sum_a \frac{r_{a_0}}{2} \iint_{S'} \frac{\vec{r}' \times \vec{n}_r (\vec{n}_r \cdot \vec{E}_{a_0}) \cdot \vec{A}'_{a_z}}{(a^2 - r'^2)^{1/2}}\, ds$$

$$- \sum_\alpha \frac{p_{\alpha_0}}{2} \iint_{S'} \frac{\vec{r}' \times \vec{n}_r (\vec{n}_r \cdot \vec{F}_{\alpha_0}) \cdot \vec{A}'_{a_z}}{(a^2 - r'^2)^{1/2}}\, ds \qquad \text{.........(5.14b)}$$

By solving the linear differential equation of Eq. (5.14a), we can determine the expansion coefficient r_{a_z} whereby s_{a_z} will be obtained

$$s_{a_z} = \frac{\partial r_{a_z}}{\partial t} \qquad \text{.........(5.15)}$$

It should be noted that Eq. (5.14b) is independent of coordinates but depends on the time dependent expansion coefficients of the fields in the main cavity. This equation is well known as a forced oscillation differential equation as it must be.

As shown in Eqs. (5.12) and (5.14a), the expansion coefficients depend on the eigenfunctions $\vec{A}'_{\alpha z}$s and $\phi_{\beta z}$s of possible modes in the subsidiary cavity. In turn, $\vec{A}'_{\alpha z}$s and $\phi_{\beta z}$s depend upon the shape, construction of the cavity, its boundary conditions, and the frequencies of the fields in the main cavity. If the shape and the boundary conditions of the subsidiary cavity are specified, the eigenfunctions $\vec{A}'_{\alpha z}$s and $\phi_{\beta z}$s will be obtained, so as the expansion coefficients $r_{\alpha z}$, $s_{\alpha z}$ and $q_{\beta z}$. Substituting these known values into Eq. (5.9) and Eq. (5.10), we shall have a complete solution for the field configuration in the subsidiary cavity. In the like manner, we shall obtain the reflected fields in the main cavity only due to the presence of the fictitious surface magnetic charge and current in the iris. Then applying the superposition method, we shall acquire the total fields in the main cavity.

The problem in determining the field configurations in a two-cavity-coupling system has been solved. In the next chapter, we shall work out an example to illustrate the application of the solutions. In the meantime, the proper procedure and technique in solving a practical problem will be given in detail. The solution obtained in the last two chapters are also applicable to the cavity-waveguide system coupled via a small circular iris.

7. EXAMPLE

The cavity-coupling system consists of a main cavity of dimensions l_{x_1}, l_{y_1} and l_{z_1}, and a subsidary cavity of dimensions l_{x_2}, l_{y_2}, and l_{z_2}. These two cavities are resonating at about the same frequency and are coupled through a circular iris of radius 'a'. The iris is located at the centre of the common wall whose thickness is negligible. The schematical diagram of this system is shown in Figure 4. Now let us assume that only TE_{301} mode of the $e^{j\omega t}$ time dependence is being excited in the main cavity. Regardless of the sources of excitation, the fields are given by:

$$S_{a_1}\vec{H}_{a_1} = -\vec{a}_x A e^{j\omega t}\left(\frac{3\pi^2}{l_{z_1} l_{x_1} k_{a_1}^2}\right)\sin\frac{3\pi x}{l_{x_1}}\cos\frac{\pi z}{l_{z_1}} + \vec{a}_z A e^{j\omega t}\left(\frac{3\pi}{l_{x_1} k_{a_1}}\right)^2 \cos\frac{3\pi x}{l_{x_1}}\sin\frac{\pi z}{l_{z_1}} \qquad \text{.........(6.1)}$$

$$g_{\beta_1}\vec{G}_{\beta_1} = \vec{a}_x B e^{j\omega t}\frac{3\pi}{l_{x_1} k_{\beta_1}}\sin\frac{3\pi x}{l_{x_1}}\cos\frac{\pi z}{l_{z_1}} + \vec{a}_z B e^{j\omega t}\frac{\pi}{l_{z_1} k_{\beta_1}}\cos\frac{3\pi x}{l_{x_1}}\sin\frac{\pi z}{l_{z_1}} \qquad \text{.........(6.2)}$$

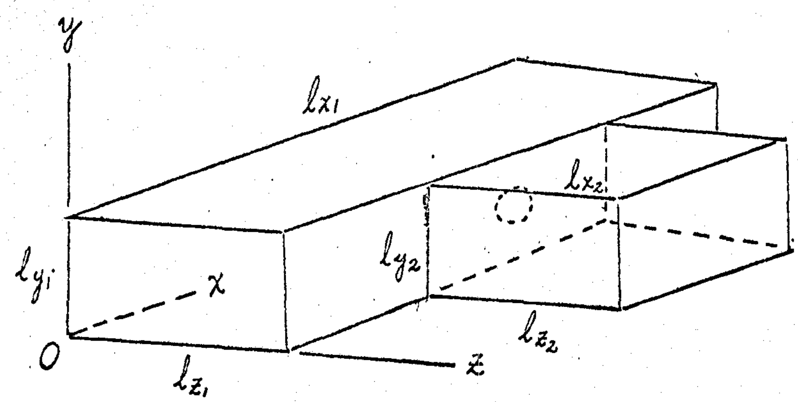

Fig. 4 Schematical diagram of the cavity-coupling system.

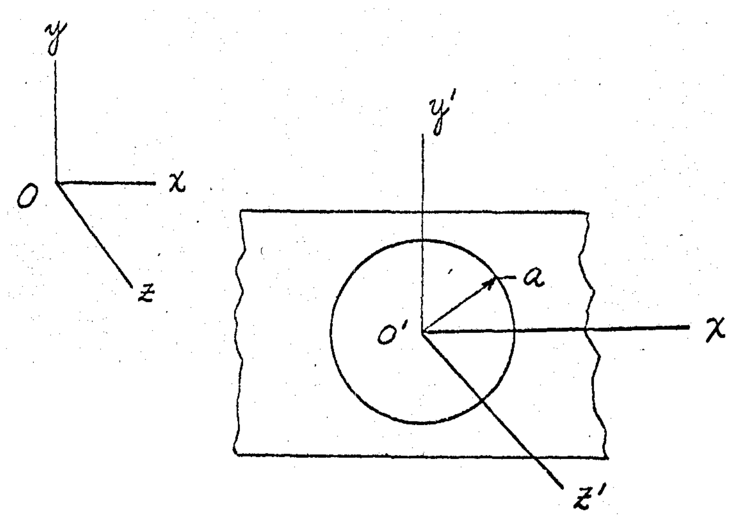

Fig. 5 New coordinated for the system.

$$\gamma_{a_1}\vec{E}_{a_1} = \vec{a}_y A e^{j\omega t}\frac{3\pi}{l_{x_1}k_{a_1}} \sin\frac{3\pi x}{l_{x_1}} \sin\frac{\pi z}{l_{z_1}} \quad \ldots\ldots\ldots(6.3)$$

where A and B are arbitrary constants and

$$k_{a_1}^2 = \left(\frac{3\pi}{l_{x_1}}\right)^2 + \left(\frac{\pi}{l_{z_1}}\right)^2 = \left(\frac{\omega}{c}\right)^2$$

$$k_{\beta_1}^2 = \left(\frac{3\pi}{l_{x_1}}\right)^2 + \left(\frac{\pi}{l_{z_1}}\right)^2 .$$

From now on, it is convenient to translate the origin of the coordinate system to a position so that it coincides with the centre of the iris and the xy-plane coincides with the plane of the iris. It is shown in Figure 5. The new coordinates (x', y', and z') and the old coordinates (x, y, and z) systems are related by

$$x = x' + l_{x_1}/2$$
$$y = y' + l_{y_1}/2$$
$$z = z' + l_{z_1}$$

With the new set of coordinates, Eqs. (6.1), (6.2), and (6.3) now become

$$s_{a_1}\vec{H}_{a_1} = \vec{a}_x A e^{j\omega t}\left(\frac{3\pi^2}{l_{z_1}l_{x_1}k_{a_1}^2}\right)\sin\left(\frac{3\pi x'}{l_{x_1}}+\frac{3\pi}{2}\right)\cos\frac{\pi z'}{l_{z_1}}$$
$$-\vec{a}_z A e^{j\omega t}\left(\frac{3\pi}{l_{x_1}k_{a_1}}\right)^2\cos\left(\frac{3\pi x'}{l_{x_1}}+\frac{3\pi}{2}\right)\sin\frac{\pi z'}{l_{z_1}}$$
$$\ldots\ldots\ldots(6.4)$$

$$q_{\beta_1}\vec{G}_{\beta_1} = -\vec{a}_x B e^{j\omega t}\left(\frac{3\pi}{l_{x_1}k_{\beta_1}}\right)\sin\left(\frac{3\pi x'}{l_{x_1}}+\frac{3\pi}{2}\right)\cos\frac{\pi z'}{l_{z_1}}$$
$$-\vec{a}_z B e^{j\omega t}\left(\frac{\pi}{l_{z_1}k_{\beta_1}}\right)\cos\left(\frac{3\pi x'}{l_{x_1}}+\frac{3\pi}{2}\right)\sin\frac{\pi z'}{l_{z_1}}$$
..........(6.5).

$$r_{a_1}\vec{E}_{a_1} = -\vec{a}_y A e^{j\omega t}\left(\frac{3\pi}{l_{x_1}k_{a_1}}\right)\sin\left(\frac{3\pi x'}{l_{x_1}}+\frac{3\pi}{2}\right)\sin\frac{\pi z'}{l_{z_1}}\,.$$
..........(6.6)

At the centre of the iris or at O'(x' = y' = z' = 0)

$$s_{a_1}\vec{H}_{a_1} = -\vec{a}_x A\left(\frac{3\pi^2}{l_{z_1}l_{x_1}k_{a_1}^2}\right)e^{j\omega t}$$
$$q_{\beta_1}\vec{G}_{\beta_1} = -a_x B\left(\frac{3\pi}{l_{x_1}k_{\beta_1}}\right)e^{j\omega t}$$
$$r_{a_1}\vec{E}_{a_1} = 0$$

Since the iris is assumed small compared to the wavelength, the fields incident on it will be considered constant throughout

$$\vec{H}_o = -\vec{a}_x\left[\frac{3A\pi^2}{l_{z_1}l_{x_1}k_{a_1}^2}+\frac{3B\pi}{l_{x_1}k_{\beta_1}}\right]e^{j\omega t}$$
..........(6.7)

$$E_o = 0$$
..........(6.8)

Knowing the fields distribution in the iris, we can determine the fictitious magnetic charges and currents by applying Eq. (5.21) and (5.22) in Section 5.

$$\eta^*(\vec{r}',t) = \frac{1}{\pi^2}\left[\frac{3A\pi^2}{l_{z_1} l_{x_1} k_{a_1}^2} - \frac{3B\pi}{l_{x_1} k_{\beta_1}}\right]\frac{\vec{a}_x\cdot\vec{r}'}{(a^2-r'^2)^{1/2}}\, e^{j\omega t} \qquad \text{.........(6.9)}$$

$$\vec{K}^*(\vec{r}',t) = -j e^{j\omega t}\left[\frac{3A\omega}{l_{z_1} l_{x_1} k_{a_1}^2} - \frac{3B\omega}{\pi l_{x_1} k_{\beta_1}}\right](a^2-r'^2)^{1/2}\,\vec{a}_x \qquad \text{.........(6.10)}$$

If we neglect the retardation, the scalar potential due to the fictitious magnetic charges is :

$$\phi(\vec{r},t) = \frac{1}{4\pi^3}\left[\frac{3A\pi^2}{l_{z_1} l_{x_1} k_{a_1}^2} - \frac{3B\pi}{l_{x_1} k_{\beta_1}}\right] e^{j\omega t}\iint_{s'}\frac{\vec{a}_x\cdot\vec{r}'}{(a^2-r'^2)^{1/2}}\,\frac{1}{|\vec{r}-\vec{r}'|}\,ds$$

After performing the integration over the iris, we have

$$\phi(\vec{r},t) = \frac{3}{8}\left[\frac{A\pi}{l_{z_1} l_{x_1} k_{a_1}^2} - \frac{B}{l_{x_1} k_{\beta_1}}\right] e^{j\omega t}\,\vec{a}_x\cdot\vec{r} \qquad \text{.........(6.11)}$$

Similarly, the vector potential due to presence of the fictitious magnetic current is :

$$\vec{A}(\vec{r},t) = -j\frac{3\omega}{8}\left[\frac{A\pi}{l_{z_1} l_{x_1} k_{a_1}^2} - \frac{B}{l_{x_1} k_{\beta_1}}\right] e^{j\omega t}\left(a^2 - \frac{r^2}{2}\right)\vec{a}_x \qquad \text{.........(6.12)}$$

The next step is to evaluate the time coefficients. From Eq. (5.11), we have :

$$q_{\beta_2} = \frac{3a^3}{\pi\mu k_{\beta_2}}\left[\frac{A\pi}{l_{x_1} l_{z_1} k_{a_1}^2} - \frac{B}{l_{x_1} k_{\beta_1}}\right]^2 e^{j\omega t} \quad \text{.........(6.13)}$$

From Eq. (5.14a) in Section 5, we have

$$\mu\epsilon \frac{\partial^2 r_{a_2}}{\partial t^2} + k_{a_2}^2 r_{a_2} = -k_{a_2}\iint_{S'} \vec{K}^* \cdot \vec{A}'_{a_2}(\vec{r}_h)\, ds \quad \text{........(6.14)}$$

Since

$$\vec{K}^* = -j\, 3\omega\left[\frac{A}{l_{x_1} l_{z_1} k_{a_1}^2} - \frac{B}{\pi l_{x_1} k_{\beta_1}}\right]\left(a^2 - \frac{r_h^2}{2}\right)\vec{a}_x$$

$$\vec{A}'_{a_2} = -j\,\frac{3\pi\omega}{8}\left[\frac{A}{l_{x_1} l_{z_1} k_{a_1}^2} - \frac{B}{\pi l_{x_1} k_{\beta_1}}\right]\left(a^2 - \frac{r_h^2}{2}\right)\vec{a}_x .$$

The integral on the right-hand side of Eq. (6.14) can be easily evaluated :

$$\iint_{S'} \vec{K}^* \cdot \vec{A}_{a_2}\, ds = -\frac{4a^5}{15}\left[\frac{3\omega}{l_{x_1}}\left(\frac{A}{l_{z_1} k_{a_1}^2} - \frac{B}{\pi k_{\beta_1}}\right)\right]^2 e^{j\omega t}.$$

By substituting this value into Eq. (6.14), we have

$$\mu\epsilon \frac{\partial^2 r_{a_2}}{\partial t^2} + k_{a_2}^2 r_{a_2} = \frac{4 k_{a_2} a^5}{15}\left[\frac{3\omega}{l_{x_1}}\left(\frac{A}{l_{z_1} k_{a_1}^2} - \frac{B}{\pi k_{\beta_1}}\right)\right]^2 e^{j\omega t}.$$

By solving this linear differential equation, we find :

$$r_{a_2} = -\frac{4\omega_{a_2} a^5 e^{j\omega t}}{15\sqrt{\mu\epsilon}\,(\omega_{a_2}^2 - \omega^2)}\left[\frac{3\omega}{l_{x_1}}\left(\frac{A}{l_{z_1} k_{a_1}^2} - \frac{B}{\pi k_{\beta_1}}\right)\right]^2\left(e^{j(\omega_{a_2}-\omega)t} - 1\right),$$

or

$$\gamma_{a_2} = - \frac{4\omega_{a_2} a^5 e^{j\omega t}}{15\sqrt{\mu\epsilon}\,(\omega_{a_2}+\omega)\Delta\omega}\left[\frac{3\omega}{l_{x_1}}\left(\frac{A}{l_{z_1}k_{a_1}^2} - \frac{B}{\pi k_{\beta_1}}\right)\right]^2 (e^{j\Delta\omega t} - 1) \,.$$

.........(6.15)

where $\Delta\omega = \omega_{a_2} - \omega$

The time coefficient for the solenoidal magnetic field is :

$$S_{a_2} = \frac{\partial \gamma_{a_2}}{\partial t}$$

$$= -j\, \frac{4\omega_a \omega a^5 e^{j\omega t}}{15\sqrt{\mu\epsilon}\,(\omega_{a_2}+\omega)\Delta\omega}\left[\frac{3\omega}{l_{x_1}}\left(\frac{A}{l_{z_1}k_{a_1}^2} - \frac{B}{\pi k_{\beta_1}}\right)\right]^2 (e^{j\Delta\omega t} - 1)$$

.........(6.16)

In case of $\omega \to \omega_{a_2}$

$$\gamma_{a_2} \doteq - \frac{2a^5 e^{j\omega t}}{15\sqrt{\mu\epsilon}\,\Delta\omega}\left[\frac{3\omega}{l_{x_1}}\left(\frac{A}{l_{z_1}k_{a_1}^2} - \frac{B}{\pi k_{\beta_1}}\right)\right]^2 (e^{j\Delta\omega t} - 1)$$

.........(6.17)

$$S_{a_2} \doteq -j\, \frac{2\omega a^5 e^{j\omega t}}{15\sqrt{\mu\epsilon}\,\Delta\omega}\left[\frac{3\omega}{l_{x_1}}\left(\frac{A}{l_{z_1}k_{a}^2} - \frac{B}{\pi k_{\beta_1}}\right)\right]^2 (e^{j\Delta\omega t} - 1)$$

.........(6.18)

Now, we assume further that only TE_{101} mode can be excited in the subsidiary cavity. The fields in the subsidary cavity are

$$s_{a_2}\vec{H}_{a_2} = j\frac{6\omega^3 a^5}{5\sqrt{\mu\epsilon}\,\Delta\omega\, l_{x_1}^2}\left(\frac{A}{l_{z_1}k_{a_1}^2} - \frac{B}{\pi k_{\beta_1}}\right)^2 e^{j\omega t}\left(e^{j\Delta\omega t} - 1\right)$$

$$\left[\vec{a}_x \frac{\pi^2}{l_{x_2}l_{z_2}k_{a_2}^2}\sin\left(\frac{\pi x'}{l_{x_2}} + \frac{\pi}{2}\right)\cos\frac{\pi z'}{l_{z_2}} - \vec{a}_z\left(\frac{\pi}{l_{x_2}k_{a_1}}\right)^2\cos\left(\frac{\pi x'}{l_{x_2}} + \frac{\pi}{2}\right)\sin\frac{\pi z'}{l_{z_2}}\right] \quad \ldots\ldots(6.19)$$

$$g_{\beta_2}\vec{G}_{\beta_2} = \frac{3a^3}{\pi\mu k_{\beta_2}}\left(\frac{A\pi}{l_{x_1}l_{z_1}k_{a_1}^2} - \frac{B}{l_{x_1}k_{\beta_1}}\right)e^{j\omega t}$$

$$\left[\vec{a}_x \frac{\pi}{l_{x_2}k_{\beta_2}}\sin\left(\frac{\pi x'}{l_{x_2}} + \frac{\pi}{2}\right)\cos\frac{\pi z'}{l_{z_2}} - \vec{a}_z\frac{\pi}{l_{z_2}k_{\beta_2}}\cos\left(\frac{\pi x'}{l_{x_2}} + \frac{\pi}{2}\right)\sin\frac{\pi z'}{l_{z_2}}\right] \quad \ldots\ldots\ldots(6.20)$$

$$r_{a_2}\vec{E}_{a_2} = \frac{6\omega^2 a^5}{5\sqrt{\mu\epsilon}\,\Delta\omega\, l_{x_1}^2}\left(\frac{A}{l_{z_1}k_{a_1}^2} + \frac{B}{\pi k_{\beta_1}}\right)^2 e^{j\omega t}\left(e^{j\Delta\omega t} - 1\right)$$

$$\left[\vec{a}_y \frac{\pi}{l_{x_2}k_{a_2}}\sin\left(\frac{\pi x'}{l_{x_2}} + \frac{\pi}{2}\right)\sin\left(\frac{\pi z'}{l_{z_2}}\right)\right]. \quad \ldots\ldots\ldots(6.21)$$

The total fields in the main cavity are now the sum of the incident fields plus the reflected fields due to the presence of the iris. Because the cavity is so designed that at frequency $\frac{\omega}{2\pi}$ only TE_{301} mode can be excited. The field configuration remains unchanged but the amplitude (or the time dependent expansion coefficients) differ slightly from the unperturbed amplitude.

$$S'_{a_1} \vec{H}_{a_1} = \left[A + j \frac{6\omega^3 a^5}{5\sqrt{\mu\epsilon}\,\Delta\omega\, l_{x_1}^2} \left(\frac{A}{l_{z_1} k_{a_1}^2} - \frac{B}{\pi k_{\beta_1}} \right)^2 \left(e^{j\Delta\omega t} - 1 \right) \right] e^{j\omega t}$$

$$\left\{ \vec{a}_x \frac{3\pi^2}{l_{x_1} l_{z_1} k_{a_1}^2} \sin\left(\frac{3\pi x'}{l_{x_1}} + \frac{3\pi}{2} \right) \cos \frac{\pi z'}{l_{z_1}} \right.$$

$$\left. - \vec{a}_z \left(\frac{\pi}{l_{x_1} k_{a_1}} \right)^2 \cos\left(\frac{3\pi x'}{l_{x_1}} + \frac{3\pi}{2} \right) \sin \frac{\pi z'}{l_{z_1}} \right\}$$

.........(6.22)

$$q'_{\beta_1} \vec{G}_{\beta_1} = \left[B + \frac{3a^3}{\pi\mu k_{\beta_1}} \left(\frac{A\pi}{l_{x_1} l_{z_1} k_{a_1}^2} - \frac{B}{l_{x_1} k_{\beta_1}} \right) \right] e^{j\omega t}$$

$$\left\{ \vec{a}_x \frac{3\pi}{l_{x_1} k_{\beta_1}} \sin\left(\frac{3\pi x'}{l_{x_1}} + \frac{3\pi}{2} \right) \cos \frac{\pi z'}{l_{z_1}} \right.$$

$$\left. - \vec{a}_z \frac{\pi}{l_{z_1} k_{\beta_1}} \cos\left(\frac{3\pi x'}{l_{x_1}} + \frac{3\pi}{2} \right) \sin \frac{\pi z'}{l_{z_1}} \right\}$$

.........(6.23)

$$Y'_{a_1} \vec{E}_{a_1} = \left[A + \frac{6\omega^2 a^5}{5\sqrt{\mu\epsilon}\,\Delta\omega\, l_{x_1}^2} \left(\frac{A}{l_{z_1} k_{a_1}^2} + \frac{B}{\pi k_{\beta_1}} \right)^2 \left(e^{j\Delta\omega t} - 1 \right) \right] e^{j\omega t}$$

$$\left\{ \vec{a}_y \frac{\pi}{l_{x_1} k_{a_1}} \sin\left(\frac{3\pi x'}{l_{x_1}} + \frac{3\pi}{2} \right) \sin \frac{\pi z'}{l_{z_1}} \right\}$$

.........(6.24)

APPENDIX

A. Vector and Scalar Functions

In any closed region V bounded by a regular surface S , we may define an arbitrary piecewise continuous vector function and an arbitrary piecewise continuous scalar function by setting up two complete sets of orthogonal eigenfunctions for each of them. Each of these satisfies the wave equation and its corresponding boundary conditions. For the vector function, we have

$$\nabla^2 \vec{\Psi}_p + k_p^2 \vec{\Psi}_p = 0 \qquad \text{(in V)} \quad \text{.........(A.1a)}$$

$$\begin{aligned} \vec{n} \times \vec{\Psi}_p &= 0 \\ \vec{n} \cdot \vec{n} \times \vec{\Psi}_p &= 0 \end{aligned} \qquad \text{(on S)} \quad \text{.........(A.1b)}$$

and

$$\nabla^2 \vec{\Phi}_q + k_q^2 \vec{\Phi}_q = 0 \qquad \text{(in V)} \quad \text{.........(A.2a)}$$

$$\begin{aligned} \vec{n} \times \nabla \times \vec{\Phi}_q &= 0 \\ \vec{n} \cdot \vec{\Phi}_q &= 0 \end{aligned} \qquad \text{(on S)} \quad \text{.........(A.2b)}$$

For the scalar function, we have

$$\nabla^2 \psi_\alpha + k_\alpha \psi_\alpha = 0 \qquad \text{(in V)........(A.3a)}$$

$$\psi_\alpha = 0 \qquad \text{(on S)........(A.3b)}$$

and

$$\nabla^2 \phi_\beta + k_\beta \phi_\beta = 0 \qquad \text{(in V)........(A.4a)}$$

$$\frac{\partial \phi_\beta}{\partial n} = 0 \qquad \text{(on S)........(A.4b)}$$

where k_p , k_q , k_α , and k_β are the eigenvalues of the eigenfunctions $\vec{\Psi}_p$, $\vec{\Phi}_q$, ψ_α , and ϕ_β respectively; p, q, α , and β are integers (= 0, 1, 2, 3, ...); $\vec{n}$ is a unit normal vector on S.

Green's theorem states that if two vector functions, $\vec{\Psi}_p$ and $\vec{\Phi}_q$, are continuous in a closed region of space V bounded by a regular surface S, we can write

$$\iiint_V [\vec{\Psi}_p \cdot (\nabla \times \nabla \times \vec{\Phi}_q) - \vec{\Phi}_q \cdot (\nabla \times \nabla \times \vec{\Psi}_p)]\, dv$$
$$= \iint_S [\vec{\Phi}_q \times \nabla \times \vec{\Psi}_p - \vec{\Psi}_p \times \nabla \times \vec{\Phi}_q] \cdot \vec{n}\, ds \qquad \text{.........(A.5)}$$

Expanding $\nabla \times \nabla \times$ terms and applying the divergence theorem, we get:

$$\iiint_V [\vec{\Psi}_p \cdot \nabla^2 \vec{\Phi}_q - \vec{\Phi}_q \cdot \nabla^2 \vec{\Psi}_p]\, dv$$
$$= \iint_S \Big\{ [\vec{\Psi}_p \nabla \cdot \vec{\Phi}_q - \vec{\Phi}_q \nabla \cdot \vec{\Psi}_p] \cdot \vec{n} - [\vec{\Psi}_p \cdot (\vec{n} \times \nabla \times \vec{\Phi}_q) + \nabla \times \vec{\Psi}_p \cdot (\vec{n} \times \vec{\Phi}_q)] \Big\}\, ds \qquad \text{.........(A.6)}$$

From Eq. (A.1a) and Eq. (A.2a), we have

$$\vec{\Psi}_p \cdot \nabla^2 \vec{\Phi}_q - \vec{\Phi}_q \cdot \nabla^2 \vec{\Psi}_p = (k_p^2 - k_q^2)\, \vec{\Psi}_p \cdot \vec{\Phi}_p$$

For $k_p = k_q$, the volume integral in Eq. (A.6) vanishes. Further applying the boundary conditions, Eq. (A.6) reduces to

$$\iint_S \left\{ (\vec{\Psi}_p \nabla \cdot \vec{\Phi}_q) \cdot \vec{n} - \nabla \times \vec{\Psi}_p \cdot (\vec{n} \times \vec{\Phi}_q) \right\} ds = \iiint_v \left[\vec{\Psi}_p \cdot \nabla^2 \vec{\Phi}_q - \vec{\Phi}_q \cdot \nabla^2 \vec{\Psi}_p \right] dv \qquad \text{.........(A.7)}$$

Since the fields are confined in an enclosed region bounded by a surface S, the integrand in Eq. (A.7) must equal to zero, i.e.,

$$(\vec{\Psi}_p \nabla \cdot \vec{\Phi}_q) \cdot \vec{n} - \nabla \times \vec{\Psi}_p \cdot (\vec{n} \times \vec{\Phi}_q) = \left[(\vec{\Psi}_p \nabla \cdot \vec{\Phi}_q) - \vec{\Phi}_q \times \nabla \times \vec{\Psi}_p \right] \cdot n = 0$$

Since none of the normal component of each term in the above expression vanishes on the surface S, it must be

$$\vec{\Psi}_p \nabla \cdot \vec{\Phi}_q - \vec{\Phi}_q \times \nabla \times \vec{\Psi}_p = 0 \qquad \text{.........(A.8)}$$

From the vector identity, we may put

$$\nabla \cdot (\vec{\Psi}_p \nabla \cdot \vec{\Phi}_q) = \nabla \cdot \vec{\Psi}_p \nabla \cdot \vec{\Phi}_q + \vec{\Psi}_p \cdot \nabla^2 \vec{\Phi}_q \qquad \text{.........(A.9)}$$

By interchanging the roles of the functions $\vec{\Phi}_q$ and $\vec{\Psi}_p$

we have

$$\nabla\cdot(\vec{\Phi}_q\,\nabla\cdot\vec{\Psi}_p) = \nabla\cdot\vec{\Phi}_q\,\nabla\cdot\vec{\Psi}_p + \vec{\Phi}_q\cdot\nabla^2\vec{\Psi}_p \qquad \text{.........(A.10)}$$

Upon substracting Eq. (A.10) from Eq. (A.9), we again get

$$\begin{aligned}&\nabla\cdot(\vec{\Psi}_p\,\nabla\cdot\vec{\Phi}_q) - \nabla\cdot(\vec{\Phi}_q\,\nabla\cdot\vec{\Psi}_p)\\ &= \vec{\Psi}_p\cdot\nabla^2\vec{\Phi}_q - \vec{\Phi}_q\cdot\nabla^2\vec{\Psi}_p\\ &= (k_p^2 - k_q^2)\,\vec{\Psi}_p\cdot\vec{\Phi}_q\end{aligned} \qquad \text{.........(A.11)}$$

The volume integral of the right hand side of Eq.(A.11) equals to zero if $k_p = k_q$. Using the divergence theorem, the volume integral of the left hand side is

$$\iiint_V[\nabla\cdot(\vec{\Psi}_p\,\nabla\cdot\vec{\Phi}_q) - \nabla\cdot(\vec{\Phi}_q\,\nabla\cdot\vec{\Psi}_p)]\,dv = \iint_S n\cdot[\vec{\Psi}_p\nabla\cdot\vec{\Phi}_q - \vec{\Phi}_q\nabla\cdot\vec{\Psi}_p]\,ds$$

or

$$\iint_S \vec{n}\cdot\vec{\Psi}_p(\nabla\cdot\vec{\Phi}_q)\,ds = \iint_S \vec{n}\cdot\vec{\Phi}_q(\nabla\cdot\vec{\Psi}_p)\,ds$$
$$= 0$$

for $\vec{n}\cdot\Phi_q = 0$ on the surface S. Since $\vec{n}\cdot\Psi_p \neq 0$ on S,

$$\vec{\Psi}_p(\nabla\cdot\vec{\Phi}_q) = 0 \qquad \text{.........(A.12)}$$

Hence, we obtain from Eq. (A.8) that

$$\vec{\Phi}_q\times\nabla\times\vec{\Psi}_p = 0 \qquad \text{.........(A.13)}$$

By the use of Eqs. (A.12), (A.2b) and (A.13), Eq. (A.5) reduces to

$$\iiint_V [\vec{\Psi}_p \cdot (\nabla \times \nabla \times \Phi_q) - \vec{\Phi}_q \cdot (\nabla \times \nabla \times \vec{\Psi}_p)]\, dv = 0$$

or

$$\vec{\Psi}_p \cdot (\nabla \times \nabla \times \vec{\Phi}_q) - \vec{\Phi}_q \cdot (\nabla \times \nabla \times \vec{\Psi}_p) = 0 \qquad \text{.........(A.14)}$$

Now let us consider Eq. (A.12) and (A.13) carefully. Since $\vec{\Psi}_p \neq \vec{\Phi}_q \neq 0$ in V or on S, Φ_q must be a curl of a vector function and $\vec{\Psi}_p$, on the other hand, may either be a gradient of a scalar function or be that its curl is proportional to Φ_q. If $\vec{\Phi}_q$ and $\vec{\Psi}_p$ also have to satisfy Eq. (A.14), the solution is

$$k_q \vec{\Phi}_q = \nabla \times \vec{\Psi}_p$$

$$k_p \vec{\Psi}_p = \nabla \times \vec{\Phi}_q$$

and $k_p = k_q$.

If we let $k_p = k_q = k_a$ $\quad (a = 1, 2, 3, \ldots)$

$$k_a \vec{\Phi}_a = \nabla \times \vec{\Psi}_a \qquad \text{.........(A.15a)}$$

$$k_a \vec{\Psi}_a = \nabla \times \vec{\Phi}_a \qquad \text{.........(A.15b)}$$

For $k_p \neq k_q$, Φ_q and $\vec{\Psi}_p$ will no longer have any correlation between them. It is quite obvious that their solutions would solely depend upon their corresponding boundary conditions which distinguishably mark their different characteristics. Let us consider the boundary condition $\vec{n} \times \vec{\Psi}_p = 0$ from Eq. (A.1b). Its surface integral will also be zero since the tangential component on S vanishes, i.e.,

$$\iint_S \vec{n} \times \vec{\Psi}_p \, ds = 0$$

But

$$\iint_S \vec{n} \times \vec{\Psi}_p \, ds = \iiint_V \nabla \times \vec{\Psi}_p \, dv = 0$$

on S or in V. Hence $\nabla \times \Psi_p = 0$ if $\vec{\Psi}_p \neq 0$. The possible solution of $\vec{\Psi}_p$ is

$$k_p \vec{\Psi}_p = \nabla \psi_p \qquad \text{.........(A.16)}$$

By using similar technique for the boundary condition $\vec{n} \times \nabla \times \Phi_q = 0$, we obtain

$$\iint_S \vec{n} \times \nabla \times \vec{\Phi}_q \, ds = \iiint_V \nabla \times \nabla \times \vec{\Phi}_q \, dv = 0$$

Now we can equate

$$\nabla \times \nabla \times \vec{\Phi}_q = 0$$

Since $\nabla \times \nabla \times = \nabla(\nabla \cdot) - \nabla^2$ and $\nabla^2 \vec{\Phi}_q + k_q^2 \vec{\Phi}_q = 0$, it becomes

$$\nabla (\nabla \cdot \vec{\Phi}_q) - k_q^2 \Phi_q = 0$$

Now we see that Φ_q is a gradient of a scalar function. If $k_q \neq 0$ and $\Phi_q \neq 0$ in V, then the solution for $\vec{\Phi}_q$

in general is

$$k_g \vec{\Phi}_g = \nabla \phi_g \qquad \text{.........(A.17)}$$

We have now proved that in a closed region bounded by a regular surface, a piecewise continuous field vector can always be expressed by an irrotational and a solenoidal function with corresponding boundary conditions. When two vector functions of different boundary conditions are presented in that closed region, one can be determined by the curl of the other for equal eigenvalues. For unequal eigenvalues, these functions are measured by a gradient of scalar functions. It is also obvious as shown in the proof that although we have defined four groups of orthogonal eigenfunctions, namely, $\vec{\Psi}_p$, $\vec{\Phi}_g$, ψ_α , and ϕ_p , the last two may be derived from the first two.

B. Orthogonality

We are satisfied that the electric field as well as the magnetic field are orthogonal, and that in a complete expansion of electromagnetic fields in a cavity resonator, they can be represented by four sets of orthogonal functions. But, for the series expansions of Eqs. (2.1) and (2.2) in Section 2 to be valid, the orthogonality of these functions must be shown. From the vector relation, we can write

$$\nabla\cdot[\vec{E}_b \times (\nabla\times\vec{E}_a)] = (\nabla\times\vec{E}_b)\cdot(\nabla\times E_a) - \vec{E}_b\cdot\nabla\times(\nabla\times\vec{E}_a) \qquad \text{.........(B.1a)}$$

$$\nabla\cdot[E_a \times (\nabla\times E_b)] = (\nabla\times\vec{E}_a)\cdot(\nabla\times E_b) - \vec{E}_a\cdot\nabla\times(\nabla\times\vec{E}_b) \qquad \text{.........(B.1b)}$$

Their difference gives

$$\nabla\cdot[\vec{E}_b \times (\nabla\times\vec{E}_a)] - \nabla\cdot[\vec{E}_a \times (\nabla\times\vec{E}_b)] = \vec{E}_a\cdot\nabla\times(\nabla\times\vec{E}_b) - \vec{E}_b\cdot\nabla\times(\nabla\times\vec{E}_a) \qquad \text{.........(B.2)}$$

Since $\nabla\times\nabla\times\vec{E}_a = \nabla(\nabla\cdot E_a) - \nabla^2\vec{E}_a$,

$$\nabla\cdot\vec{E}_a = 0\ ,$$

$$\nabla^2\vec{E}_a + k_a^2\,\vec{E}_a = 0$$

Eq. (B.2) reduces to

$$\nabla\cdot[\vec{E}_b\times(\nabla\times\vec{E}_a)] - \nabla\cdot[\vec{E}_a\times(\nabla\times\vec{E}_b)] = (k_a^2 - k_b^2)\,\vec{E}_a\cdot\vec{E}_b$$

Upon taking volume integral on both sides and using the divergence theorem, we obtain

$$(k_a^2 - k_b^2)\iiint_V \vec{E}_a\cdot\vec{E}_b\,dv = \iint_S \{\vec{n}\cdot[\vec{E}_b\times(\nabla\times\vec{E}_a) - \vec{E}_a\times(\nabla\times\vec{E}_b)]\}ds$$

$$= k_a\iint_S \vec{n}\cdot[\vec{E}_b\times\vec{H}_a - \vec{E}_a\times\vec{H}_b]\,ds \qquad \text{.........(B.3)}$$

Owing to the vector identities

$$\vec{n}\cdot \vec{E}_a \times \vec{H}_b = \vec{H}_b \cdot (\vec{n} \times \vec{E}_a)$$
$$n\cdot \vec{E}_b \times \vec{H}_a = \vec{H}_a \cdot (\vec{n} \times \vec{E}_b)$$

and the boundary conditions, the surface integral in Eq. (B.3) equals to zero. If $a \neq b$, $k \neq k$,

$$\iiint_V \vec{E}_a \cdot \vec{E}_b \, dv = 0 \qquad \text{.........(B.4)}$$

If $a = b$, Eq. (B.1a) or Eq. (B.1b) becomes

$$\nabla \cdot [\vec{E}_a \times (\nabla \times \vec{E}_a)] = (\nabla \times \vec{E}_a)\cdot(\nabla \times \vec{E}_a) - \vec{E}_a \cdot \nabla \times (\nabla \times \vec{E}_a)$$

From the relations of Eqs. (2.5a) and (2.5b), we have

$$\nabla \cdot [\vec{E}_a \times (\nabla \times \vec{E}_a)] = k_a^2 H_a^2 - k_a^2 E_a^2$$

As before, the volume integral of the left hand side of the above expression vanishes. The volume integral of the right hand side is

$$k_a^2 \iiint_V (H_a^2 - E_a^2)\, dv = 0$$

Hence,

$$\iiint_V H_a^2 \, dv = \iiint_V E_a^2 \, dv = 0 \qquad \text{.........(B.5)}$$

where C is an arbitrary constant. Through the similar procedure as above, we have the similar results for the functions $\vec{H}_a$ and $\vec{H}_b$. After the normalization, we finally have

$$\iiint_V \vec{E}_a \cdot \vec{E}_b \, dv = \delta_{ab} \qquad \text{.........(B.6)}$$

$$\iiint_V \vec{H}_a \cdot \vec{H}_b \, dv = \delta_{ab} \qquad \text{.........(B.7)}$$

and

$$\delta_{ab} = \begin{cases} 0 & \text{if } a \neq b \\ 1 & \text{if } a = b \end{cases}$$

Now, let us consider

$$\nabla \cdot (\psi_b \nabla \psi_a) = \nabla \psi_a \cdot \nabla \psi_b - \psi_a \nabla^2 \psi_b$$

Because of Eq. (2.6b), it can be rewritten as

$$\nabla \cdot (\psi_a \nabla \psi_b) = \nabla \psi_a \cdot \nabla \psi_b - k_b^2 \psi_a \psi_b \qquad \text{.........(B.8)}$$

By interchanging the roles of ψ_a and ψ_b , we again get

$$\nabla \cdot (\psi_b \nabla \psi_a) = \nabla \psi_a \cdot \nabla \psi_b - k_a^2 \psi_a \psi_b \qquad \text{.........(B.9)}$$

The substraction of Eq. (B.8) from Eq. (B.9) gives

$$(k_a^2 - k_b^2) \psi_a \psi_b = \nabla \cdot (\psi_a \nabla \psi_b) - \nabla \cdot (\psi_b \nabla \psi_a)$$

Again taking volume integration on both sides and using the divergence theorem on the right hand side, we have

$$(k_a^2 - k_b^2) \iiint_V \psi_a \psi_b \, dv = \iiint_V [\nabla \cdot (\psi_a \nabla \psi_b) - \nabla \cdot (\psi_b \nabla \psi_a)] dv$$

$$= \iint_S [\psi_a \vec{n} \cdot \nabla \psi_b - \psi_b \vec{n} \cdot \nabla \psi_a] \, ds$$

Since $\psi_a = \psi_b = 0$ on S,

$$(k_a^2 - k_b^2) \iiint_V \psi_a \psi_b \, dv = 0$$

If $a \neq b$, $k_a \neq k_b$, then

$$\iiint_V \psi_a \psi_b \, dv = 0 \qquad \text{.........(B.10)}$$

From Eqs. (B.8) and (2.6a), we obtain

$$k_b \nabla \cdot (\psi_a \vec{F}_b) = k_b (k_a \vec{F}_a \cdot \vec{F}_b - k_b \psi_a \psi_b)$$

Using the same integration technique as before,

$$\iiint_V \nabla \cdot (\psi_a \vec{F}_b) \, dv = k_a \iiint_V \vec{F}_a \cdot \vec{F}_b \, dv - k_b \iiint_V \psi_a \psi_b \, dv$$

$$\iint_S \vec{n} \cdot \vec{F}_b \psi_a \, ds = k_a \iiint_V \vec{F}_a \cdot \vec{F}_b \, dv - k_b \iiint_V \psi_a \psi_b \, dv$$

For the boundary condition $\psi_a = 0$ on S,

$$k_a \iiint_V \vec{F}_a \cdot \vec{F}_b \, dv - k_b \iiint_V \psi_a \psi_b \, dv = 0$$

If $a \neq b$, $\iiint_V \psi_a \psi_b \, dv = 0$ as before. Hence,

$$\iiint_V \vec{F}_a \cdot \vec{F}_b \, dv = 0 \qquad \text{.........(B.11)}$$

If $a = b$,

$$\iiint_V F_a^2 \, dv = \iiint_V \psi_a^2 \, dv = C$$

After normalization,

$$\iiint_V \vec{F}_a \cdot \vec{F}_b \, dv = \delta_{ab} \qquad \text{.........(B.12)}$$

$$\iiint_V \psi_a \psi_b \, dv = \delta_{ab} \qquad \text{..........(B.13)}$$

and

$$\delta_{ab} = \begin{cases} 0 & \text{if } a \neq b \\ 1 & \text{if } a = b \end{cases}$$

Similarly, we can find

$$\iiint_V \vec{G}_a \cdot \vec{G}_b \, dv = \delta_{ab} \qquad \text{..........(B.14)}$$

$$\iiint_V \phi_a \cdot \phi_b \, dv = \delta_{ab} \qquad \text{..........(B.15)}$$

and

$$\delta_{ab} = \begin{cases} 0 & \text{if } a \neq b \\ 1 & \text{if } a = b \end{cases}$$

Again let us put

$$\nabla \cdot (\psi_a \vec{E}_b) = \psi_a \nabla \cdot \vec{E}_b + \nabla \psi_a \cdot \vec{E}_b = k_a \vec{F}_a \cdot \vec{E}_a .$$

The combination of the use of volume integration and divergence theorem yields

$$\iiint_V \nabla \cdot (\psi_a \vec{E}_b) \, dv = \iint_S \vec{n} \cdot \vec{E}_b \psi_a \, ds$$
$$= k_a \iiint_V \vec{F}_a \cdot E_b = 0$$

for $\psi_a = 0$ on S. Therefore,

$$\iiint_V \vec{E}_b \cdot \vec{F}_a \, dv = 0 \qquad \text{..........(B.16)}$$

for all a and b. Similarly, we can prove that

$$\iiint_V \vec{H}_b \cdot \vec{G}_a \, dv = 0 \qquad \text{.........(B.17)}$$

for all a and b.

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